Thursday Quiz.

1. What is your favorite automobile type?

2. Define closed set.

3. State the Cauchy Criterion for sequences.

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4. We say that the sequence (a_n) is __________ if $a_{n+1} \leq a_n \ \forall n \in \mathbb{N}$.

From last time:

Exercise 3.2.2. Let

$$A = \left\{ (-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots \right\} \quad \text{and} \quad B = \{ x \in \mathbf{Q} : 0 < x < 1 \} .$$

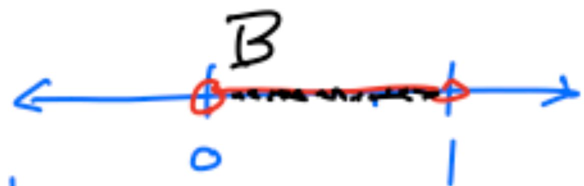
Answer the following questions for each set:

(a) What are the limit points?

(b) Is the set open? Closed?

(c) Does the set contain any isolated points?

(d) Find the closure of the set.



(a) $L_A = \{-1, 1\}$ $\qquad L_B = [0, 1]$

every # in here is a limit of a sequence in B.

(b) Is the set open? closed?

A is not closed, b/c -1 is a limit pt, but $-1 \notin A$.

A is not open, b/c e.g. $2 \in A$, but $\forall \varepsilon > 0$ $V_\varepsilon(2)$ contains a point of the form $2 + \frac{\varepsilon}{2}$, which is not in A.

So there is no $V_\varepsilon(z)$ that is contained in A.

B is not closed because $0 \in L_B$, $0 \notin B$.

B is also not open. For any $q \in B$, let $\varepsilon > 0$

$$V_\varepsilon(q) = (q-\varepsilon, q+\varepsilon).$$

So $q, q+\frac{\varepsilon}{2}$ are two real #s and $q < q+\frac{\varepsilon}{2}$. But there exists $r \in \mathbb{Q}^c$ s.t. $q < r < q+\frac{\varepsilon}{2}$.

So $V_\varepsilon(q) \not\subseteq B$.

(We are using this lemma:

<u>Lemma</u>. Let $a < b$, where $a, b \in \mathbb{R}$.

Then (a) $\exists\, q \in \mathbb{Q}$ s.t. $a < q < b$.

(b) $\exists\, r \in \mathbb{Q}^c$ s.t. $a < r < b$.

A way to construct r

let $r = q + \frac{\sqrt{2}}{n} < b$ for a large n.

(by Archie $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < \frac{b-q}{\sqrt{2}}$).

(Lemma: $\overline{\mathbb{Q}} = \mathbb{R}$.)
($\mathbb{Q}$ is <u>dense</u> in $\mathbb{R}$)

(c) Does the set contain any isolated pts?

Exercise 3.2.2. Let

$$A = \left\{ (-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots \right\} \quad \text{and} \quad B = \{ x \in \mathbf{Q} : 0 < x < 1 \}.$$

Answer the following questions for each set:

(a) What are the limit points?

(b) Is the set open? Closed?

(c) Does the set contain any isolated points?

(d) Find the closure of the set.

<u>A</u>:

Yes, all points of A besides 1 are isolated pts. (All points of A are either limit pts or isolated pts. Since the only limit pts of A are 1 and -1, (and $-1 \notin A$), all other points are isolated.

<u>B</u>: Since $L_B = [0,1]$ and $B \subseteq [0,1]$, there are no isolated points.

(d) Find the closure of the set.

$\overline{A} = A \cup L_A = A \cup \{-1\}$.

$\overline{B} = B \cup L_B = (\mathbb{Q} \cap (0,1)) \cup [0,1] = [0,1]$.

Exercise 3.2.4. Let A be nonempty and bounded above so that $s = \sup A$ exists.

(a) Show that $s \in \overline{A}$.

(b) Can an open set contain its supremum?

(a) Pretend we did it. ✓ $\sup A \in \overline{A}$.

(b) Can an open set contain its sup?

Let $U \subseteq \mathbb{R}$ be open.

Let $x = \sup U$. (so U is bounded from above.)

Suppose $x \in U$. Then, since U is open, $\exists \varepsilon > 0$ s.t. $V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) \subseteq U$.

But then $x + \frac{\varepsilon}{1.8} \in V_\varepsilon(x) \subseteq U$.

but $x + \frac{\varepsilon}{1.8} > x$, so x is not an upperbound for U.

Thus, the assumption $x \in U$ is false.

So the $\sup U$ is never in U!

<u>Definition</u>: A set $A \subseteq \mathbb{R}$ is <u>compact</u> if it is ~~nonempty and~~ every sequence of points in A has a subsequence of points converging to a point of A.

<u>Equivalently</u> A set $A \subseteq \mathbb{R}$ is <u>compact</u> if every seq of points of A contains a <u>Cauchy subsequence converging to a point of A.</u>

<u>examples</u>:

① A closed interval is compact.

<u>Pf</u>: Let $[a,b]$ be a closed interval, and suppose (x_j) is a sequence in $[a,b]$, i.e. s.t. $a \leq x_j \leq b$ $\forall j$. By BW, $\exists$ subsequence (x_{j_k}) that converges.

Since $a \leq x_{j_k} \leq b$, $a \leq \lim_{k \to \infty} x_{j_k} \leq b$ by OLT. So $\lim x_{j_k} \in [a,b]$.

Thus, $[a,b]$ is compact. $\boxtimes$

(b) The interval $[0,\infty)$ is closed but not compact.

<u>Proof</u>: Suppose L is a limit pt of $[0,\infty)$, so that $\exists$ seq. (x_j) inside $[0,\infty)$, such that $\lim x_j = L$. Then $x_j \geq 0$ $\forall j$, (where $x_j \neq L$ $\forall j$)

so $L = \lim x_j \geq 0$ by OLT.

$\therefore$ $L \in [0,\infty)$. Thus $[0,\infty)$ contains its limit pts and is thus closed. ✓

Consider the sequence $(1,2,3,\ldots) = (n)$ of pts in $[0,\infty)$. Every subsequence of this sequence satisfies (n_k)

with $n_k \geq k$, so

it is also unbounded & thus diverges.

$\therefore$ $[0,\infty)$ is not compact. ∎

<u>Thm</u>. A set $C \subseteq \mathbb{R}$ is compact

$\Longleftrightarrow$ it is closed & bounded.

Nested Compact Sets property.

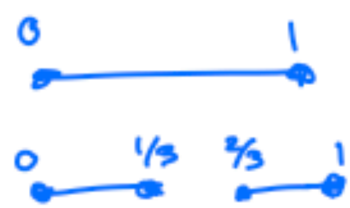
Let $\{C_i\}$ a nested sequence of nonempty compact sets, so that

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$$

Then $\bigcap_{j=1}^{\infty} C_j$ is a nonempty compact set.

Important Example – The Cantor Set

A sequence of nested nonempty compact sets:



$$C_0 = [0,1]$$

$$C_1 = [0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1]$$

$$C_2 = [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1]$$

In general

$$C_{k+1} = C_k \setminus \{\text{middle third open intervals}\}$$

The cantor Set $= C = C_\infty = \bigcap_{j=0}^{\infty} C_j$ ← a compact set.

Cardinality of C = Cardinality of $[0,1]$.

$$C \xleftrightarrow{\text{1-1 correspn}} [0,1].$$

But - contains no intervals.

C is uncountable